Multiple and partial regression and correlation

Partial $r^2$, contribution and fraction $[a]$

Daniel Borcard
Université de Montréal
Département de sciences biologiques
January 2002

The definitions of the three following notions are often confusing:

1. Contribution of a variable $x_j$ to the explanation of the variation of a dependent variable $y$.
2. Fraction $[a]$ in variation partitioning.
3. Partial $r^2$ (partial determination coefficient) between an $x_j$ and a $y$ variable.

This document assumes that the bases of simple and multiple regression, as well as the principle of the variation partitioning in regression (see Legendre & Legendre (1998), p. 528 sq.) are known to the reader.

Let us first define the three notions.

Definitions

1. **Contribution** of a variable $x_j$ to the explanation of the variation of a dependent variable $y$;

This expression is used by Scherrer (1984) in the framework of the computation of the coefficient of multiple determination in multiple regression (p. 699-700). Note that Scherrer inadequately calls "coefficient de corrélation multiple" (multiple correlation coefficient) the coefficient of multiple determination ($R^2$). The multiple correlation coefficient ($R$) is the square root of the coefficient of multiple determination.

The coefficient of multiple determination measures the proportion of the variance of a dependent variable $y$ explained by a set of explanatory variables $x_{p-1}$. It can be computed as

$$R^2 = \sum_{j=1}^{k} a'_j r_{yx_j}$$

(eq. 1)

where $a'_j$ is the standardized regression coefficient of the $j$-th explanatory variable and $r_{yx_j}$ is the simple correlation coefficient (Pearson $r$) between $y$ and $x_j$.

---

1 In Scherrer's notation: $R^2 = \sum_{j=1}^{p-1} a'_j r_{jp}$, where the $p$-th variable is the dependent variable.
In this context, Scherrer calls the quantity \( a_j r_{yx_j} \) the "contribution" of the \( j \)-th variable to the explanation of the variance of \( y \). The sum of the contributions of all explanatory variables \( x_k \) gives \( R^2 \). A contribution can be positive or negative.

2. **Fraction [a]** in variation partitioning

This fraction measures the proportion of the variance of \( y \) explained by the explanatory variable \( x_j \) (for example) when the other explanatory variables \( (x_2, x_3, ...) \) are held constant with respect to \( x_j \) only (and not with respect to \( y \)).

Thus, one obtains fraction [a] by examining the \( r^2 \) obtained by regressing \( y \) on the residuals of a regression of \( x_j \) on \( x_2, x_3, ... \)

Note: one can compute fraction [a] for several explanatory variables simultaneously, but for the sake of simplicity we illustrate here the case where one seeks the fraction with respect to a single explanatory variable only.

3. **Partial \( r^2 \)** (coefficient of partial determination) between an \( x_j \) and a \( y \) variable.

Note: when only two variables are involved, one generally uses the lowercase for \( r \) or \( r^2 \). Uppercase \( R \) and \( R^2 \) are used for the coefficients of multiple correlation and determination respectively.

First, the partial \( r \) measures the mutual relationship between two variables \( y \) and \( x \) when other variables \( (x_1, x_2, x_3, ...) \) are held constant with respect to the two variables involved \( y \) and \( x_j \) (contrary to the previous case)

The partial correlation coefficient is very useful in multiple regression, where it "...allows to directly estimate the proportion of unexplained variation of \( y \) that becomes explained with the addition of variable \( x_j \) [to the model]" (translated from Scherrer, p.702).

For an explanatory variable \( x_1 \) and a variable \( x_2 \) held constant, the partial correlation coefficient is computed as follows (Scherrer, eq. 18-50, p. 704):

\[
 r_{y,x_1|x_2} = \frac{r_{y,x_1} - r_{y,x_2} r_{x_1,x_2}}{\sqrt{(1-r_{y,x_2}^2)(1-r_{x_1,x_2}^2)}} \quad \text{(eq. 2)}
\]

The partial \( r^2 \) is the square of the partial \( r \) above. It measures the proportion of the variance of the residuals of \( y \) with respect to \( x_2 \) that is explained by the residuals of \( x_1 \) with respect to \( x_2 \). It can thus also be obtained by examining the \( r^2 \) of a regression of the residuals of \( y \) with respect to \( x_2 \) on the residuals of \( x_1 \) with respect to \( x_2 \).

Let us now examine the properties of these three measures in two different situations, each implying a dependent variable \( y \) and two explanatory variables \( x_1 \) and \( x_2 \).
Example 1. Intercorrelated (i.e., collinear) explanatory variables (most general case)

Data:

<table>
<thead>
<tr>
<th>y</th>
<th>x₁</th>
<th>x₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.000</td>
<td>1.000</td>
<td>8.000</td>
</tr>
<tr>
<td>2.000</td>
<td>1.000</td>
<td>7.000</td>
</tr>
<tr>
<td>3.000</td>
<td>1.000</td>
<td>7.000</td>
</tr>
<tr>
<td>4.000</td>
<td>1.000</td>
<td>9.000</td>
</tr>
<tr>
<td>5.000</td>
<td>1.000</td>
<td>5.000</td>
</tr>
<tr>
<td>5.000</td>
<td>1.000</td>
<td>4.000</td>
</tr>
<tr>
<td>7.000</td>
<td>2.000</td>
<td>3.000</td>
</tr>
<tr>
<td>5.000</td>
<td>2.000</td>
<td>6.000</td>
</tr>
<tr>
<td>4.000</td>
<td>2.000</td>
<td>7.000</td>
</tr>
<tr>
<td>9.000</td>
<td>2.000</td>
<td>2.000</td>
</tr>
<tr>
<td>7.000</td>
<td>2.000</td>
<td>3.000</td>
</tr>
<tr>
<td>6.000</td>
<td>2.000</td>
<td>2.000</td>
</tr>
</tbody>
</table>

Simple linear correlation matrix:

\[
\begin{pmatrix}
4.000 & 0.677 & -0.824 \\
0.677 & 1.000 & -0.612 \\
-0.824 & -0.612 & 1.000 \\
\end{pmatrix}
\]

Partial linear correlation matrix:

\[
\begin{pmatrix}
4.000 & 0.385 & -0.704 \\
0.385 & 1.000 & -0.131 \\
-0.704 & -0.131 & 1.000 \\
\end{pmatrix}
\]

Regression coefficients:

<table>
<thead>
<tr>
<th></th>
<th>Raw coeff.</th>
<th>Standardized coeff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>6.300</td>
<td>0.000</td>
</tr>
<tr>
<td>x₁</td>
<td>1.019</td>
<td>0.276</td>
</tr>
<tr>
<td>x₂</td>
<td>-0.523</td>
<td>-0.655</td>
</tr>
</tbody>
</table>

The coefficient of multiple determination of the regression can be computed using equation 1 (see above):

\[
R^2 = (0.276 \times 0.677) + (-0.655 \times -0.824) = 0.187 + 0.540 = 0.727
\]

The intermediate calculations yield the following informations:

- contribution of x₁ to the explanation of the variance of y = 0.187
- contribution of x₂ to the explanation of the variance of y = 0.540

Partial \( r^2 \) of y and x₁, holding x₂ constant with respect to y and x₁ (equation 2):

\[
r_{y,x_1|x_2}^2 = \frac{0.677 - (-0.824 \times -0.612)}{\sqrt[2]{1 - (-0.824)^2}} = \frac{0.677 - (-0.824 \times -0.612)}{\sqrt[2]{0.3210 \times 0.6255}} = \frac{0.1727}{0.4481} = 0.385
\]

\[
r_{y,x_1|x_2}^2 = 0.385^2 = 0.148
\]
Partial $r^2$ of $y$ and $x_2$, holding $x_1$ constant with respect to $y$ and $x_2$:

$$r_{y,x_2|x_1} = \frac{-0.824-(0.677 \times -0.612)}{\sqrt{[1-(0.677)^2][1-(-0.612)^2]}} = \frac{-0.4097}{\sqrt{0.5417 \times 0.6255}} = \frac{-0.4097}{0.5821} = -0.704$$

$$r^2_{y,x_2|x_1} = -0.704^2 = 0.496$$

**Fraction [a]:** proportion of the variance of $y$ explained by the explanatory variable $x_1$ when $x_2$ is held constant with respect to $x_1$ only (and not with respect to $y$). This is the $r^2$ obtained by regressing $y$ on the residuals of a regression of $x_1$ on $x_2$ (details omitted):

$$[a] = 0.0476$$

**Fraction [c]:** proportion of the variance of $y$ explained by the explanatory variable $x_2$ when $x_1$ is held constant with respect to $x_2$ only (and not with respect to $y$). This is the $r^2$ obtained by regressing $y$ on the residuals of a regression of $x_2$ on $x_1$ (details omitted):

$$[c] = 0.268$$

**Fraction [b]:** $R^2$ of the multiple regression of $y$ on $x_1$ and $x_2$ – [a] – [c] = 0.727 – 0.048 – 0.268 = 0.411

Adding fractions [a], [b], [c] et[d] gives:

$$0.048 + 0.411 + 0.268 + (1-0.727) = 0.048 + 0.411 + 0.268 + 0.273 = 1.000$$

Note: using these results, one can also calculate the partial $r^2$ of $y$ on $x_1$, controlling for $x_2$; this partial $r^2$ can indeed be construed as $[a]/[a]+[d]$:

$$r^2_{y,x_1|x_2} = 0.048/(0.048+0.273) = 0.148$$

The same calculation could be done for the partial $r^2$ of $y$ on $x_2$. 


Example 2. Explanatory variables orthogonal (linearly independent) with respect to one another

Variables $x_1$ and $x_2$ can represent, for instance, two classification criteria describing an experimental design with two orthogonal factors. In that case, the analysis of variance can be computed by multiple regression, as in this example.

**Data:**

<table>
<thead>
<tr>
<th></th>
<th>$y$</th>
<th>$x_1$</th>
<th>$x_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.000</td>
<td>1.000</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>2.000</td>
<td>1.000</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>3.000</td>
<td>1.000</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>4.000</td>
<td>1.000</td>
<td>2.000</td>
<td></td>
</tr>
<tr>
<td>5.000</td>
<td>1.000</td>
<td>2.000</td>
<td></td>
</tr>
<tr>
<td>5.000</td>
<td>1.000</td>
<td>2.000</td>
<td></td>
</tr>
<tr>
<td>7.000</td>
<td>2.000</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>5.000</td>
<td>2.000</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>4.000</td>
<td>2.000</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>9.000</td>
<td>2.000</td>
<td>2.000</td>
<td></td>
</tr>
<tr>
<td>7.000</td>
<td>2.000</td>
<td>2.000</td>
<td></td>
</tr>
<tr>
<td>6.000</td>
<td>2.000</td>
<td>2.000</td>
<td></td>
</tr>
</tbody>
</table>

**Simple linear correlation matrix:**

<table>
<thead>
<tr>
<th></th>
<th>$y$</th>
<th>$x_1$</th>
<th>$x_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>0.677</td>
<td>0.496</td>
<td></td>
</tr>
<tr>
<td>$x_1$</td>
<td>0.000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Partial linear correlation matrix:**

<table>
<thead>
<tr>
<th></th>
<th>$y$</th>
<th>$x_1$</th>
<th>$x_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>0.780</td>
<td>0.674</td>
<td></td>
</tr>
<tr>
<td>$x_1$</td>
<td>-0.526</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Regression coefficients:**

<table>
<thead>
<tr>
<th></th>
<th>Raw coeff.</th>
<th>Standardized coeff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>-1.417</td>
<td>0.000</td>
</tr>
<tr>
<td>$x_1$</td>
<td>2.500</td>
<td>0.677</td>
</tr>
<tr>
<td>$x_2$</td>
<td>1.833</td>
<td>0.496</td>
</tr>
</tbody>
</table>

Coefficient of **multiple determination** (equation 1):

\[
R^2 = (0.677 \times 0.677) + (0.496 \times 0.496) = 0.458 + 0.246 = 0.704
\]

Therefore:

- **contribution** of $x_1$ to the explanation of the variance of $y = 0.458$
- **contribution** of $x_2$ to the explanation of the variance of $y = 0.246$

**Partial $r^2$** of $y$ and $x_1$, holding $x_2$ constant with respect to $y$ and $x_1$ (equation 2):

\[
r_{y,x_1|x_2}^2 = \frac{0.677 - 0.496 \times 0}{\sqrt{(1 - 0.496^2)(1 - 0^2)}} = \frac{0.677}{\sqrt{0.754 \times 1}} = \frac{0.677}{0.868} = 0.780
\]

\[r_{y,x_1|2}^2 = 0.780^2 = 0.608\]
Partial $r^2$ of $y$ and $x_2$, holding $x_1$ constant with respect to $y$ and $x_2$:

$$r^2_{y,x_1|x_2} = \frac{0.496 - 0.677 \times 0}{\sqrt{(1 - 0.677^2)(1 - 0^2)}} = \frac{0.496}{\sqrt{0.542 \times 1}} = \frac{0.496}{0.736} = 0.674$$

$$r^2_{y,x_1|x_2} = 0.674^2 = 0.454$$

Fraction [a]: proportion of the variance of $y$ explained by the explanatory variable $x_1$ when $x_2$ is held constant with respect to $x_1$ only (and not with respect to $y$). In this example, $x_1$ and $x_2$ are orthogonal (linearly independent), so that fraction [a] is directly equal to the $r^2$ obtained by regressing $y$ on $x_1$ (details omitted):

$$[a] = 0.458$$

Fraction [c]: proportion of the variance of $y$ explained by the explanatory variable $x_2$ when $x_1$ is held constant with respect to $x_2$ only (and not with respect to $y$). In this example, $x_1$ and $x_2$ are orthogonal (linearly independent), so that fraction [c] is directly equal to the $r^2$ obtained by regressing $y$ on $x_2$ (details omitted):

$$[c] = 0.246$$

Fraction [b]: $R^2$ of the multiple regression of $y$ on $x_1$ and $x_2$ – [a] – [c] =

$$= 0.704 - 0.458 - 0.246 = 0.000$$

The orthogonality of the explanatory variables $x_1$ and $x_2$ translates into a fraction [b] equal to 0.

Adding fractions [a], [b], [c] et[d] gives:

$$0.458 + 0.000 + 0.246 + (1 - 0.704) = 0.458 + 0.000 + 0.246 + 0.296 = 1.000$$

In this case (orthogonality of $x_1$ and $x_2$), fractions [a] and [c] are equal to the contributions of variables $x_1$ and $x_2$ to the explanation of the variance of $y$.

Here again one can verify that equation $r^2_{y,x_1|x_2} = [a]/[a]+[d]$ gives the partial $r^2$ obtained above:

$$r^2_{y,x_1|x_2} = 0.458/(0.458+0.296) = 0.607 = 0.608$$

Comments (and summary of the computations)

1. Contribution (sensu Scherrer) of an explanatory variable is equal to fraction [a] of the explained variation (in the sense of a variation partitioning) in one case only: when all the explanatory variables are orthogonal to each other (linearly independent, uncorrelated). In that case, fraction [b] of the variation partitioning is equal to zero, since each explanatory variable explains a completely different fraction of the variance of $y$.

The total $R^2$ of the multiple regression (coefficient of multiple determination) is then computed as follows:

- either by summing the $a_j r_{y,x_j}$
- or by summing fractions [a] and [c] of the partitioning (since [b] equals zero!).
2. In the general case, i.e., when the explanatory variables are more or less intercorrelated (collinear), each one explains a fraction of the variation of \( y \), but these fractions overlap more or less. Each explanatory variable "does a part of the job" of the other(s), since they are partly intercorrelated. The result is a non-zero fraction \([b]\). In general, fraction \([b]\) is positive, and "nibbles" thus a part of fractions \([a]\) and \([c]\). Therefore, \([a]\) and \([c]\) are smaller than their partial contributions which comprise fraction \([a]\) or \([c]\) plus a part of fraction \([b]\).

In that case, the total \( R^2 \) of the multiple regression (coefficient of multiple determination) is computed as follows:
- either by summing the \( a_j r_{y,x_j} \)
- or by summing fractions \([a]\), \([c]\) and \([b]\) (since the latter does not equal zero).

Note: negative fractions \([b]\) are sometimes observed! This happens when two explanatory variables have strong and opposite effects of the dependent variable, while being strongly intercorrelated. In that case, fractions \([a]\) and \([c]\) are larger than their partial contributions! A detailed explanation of this pattern is given by Legendre & Legendre (1998, p. 533).

**Computation**

To obtain, for example, fraction \([a]\) of a regression of \( y \) explained by \( x_1 \) and \( x_2 \), you proceed as follows:

Step 1: compute a regression of \( x_1 \) explained by \( x_2 \) and keep the residuals. By doing this you have removed the effects of the other explanatory variable \( (x_2) \) from the explanatory variable of interest \( (x_1) \).

Step 2: compute a regression of \( y \) explained by the residuals obtained above. The \( r^2 \) of this second regression is equal to fraction \([a]\). Hence, you have explained \( y \) with the part of \( x_1 \) that has no relationship with \( x_2 \).

If you are not interested in obtaining the fitted values, but only in the values of the fractions, you can perform the whole partitioning without resorting to partial regression. The steps are the following:

1. Regression of \( y \) on \( x_1 \) yields \([a]\) + \([b]\).
2. Regression of \( y \) on \( x_2 \) yields \([b]\) + \([c]\).
3. Multiple regression of \( y \) on \( x_1 \) and \( x_2 \) yields \([a]\) + \([b]\) + \([c]\).
4. \([a]\) is obtained by subtracting the result of step 2 from that of step 3.
5. \([b]\) is obtained by subtracting the result of step 3 from that of step 1.
6. \([d]\) is obtained by subtracting from the value 1.0 the result of step 3.

3. As explained above, the partial \( r^2 \) measures the (square of the) mutual relationship between two variables when other variables are held constant with respect to both variables involved. Remember that model I regression quantifies the effect of an explanatory variable on a dependent variable, while correlation measures their mutual relationship. This is also true in the partial case. There are two possible ways of computing it. The steps below show the computation of a partial \( r^2 \) between \( y \) and an \( x_1 \) variable, removing the effect of a variable \( x_2 \) as in the case of fraction \([a]\), while emphasising the differences between the two methods:
• Computation using residuals

1. Compute a regression of $y$ explained by $x_2$ and keep the residuals. The effect of $x_2$ has been removed from $y$.

2. Compute a regression of $x_1$ explained by $x_2$ and keep the residuals. The effect of $x_2$ has been removed from $x_1$.

3a. First variant: run a regression of the residuals of step 1 explained by the residuals of step 2 above.

3b. Second variant: compute the linear correlation between the residuals of step 1 and the residuals of step 2 above. This is another way of computing the partial correlation coefficient.

Contrary to the example given above for fraction [a], this operation completely evacuates the influence of $x_2$, from $y$ as well as from $x_1$. Consequently, the $r^2$ obtained is the partial $r^2$ of $y$ and $x_1$, controlling for $x_2$! The standardized slope (standardized regression coefficient) of this simple linear regression is the partial correlation coefficient between $y$ and $x_1$.

Note that, if $x_1$ and $x_2$ are orthogonal, step 2 is useless, since $x_2$ explains nothing of $x_1$. Therefore, the residuals are equal to (centred) $x_1$.

• Computation through the fractions of variation

The partial $r^2$ can also be computed as $[a]/([a]+[d])$. Examination of this equation shows that one compares:

- in the numerator, the fraction of the variance of $y$ explained only by $x_1$;
- in the denominator, this same fraction plus the unexplained variance, but not fractions [b] and [c] corresponding to the effect of variable $x_2$.

4. Finally, I remind the reader that fraction [b] has nothing to do with an interaction in ANOVA. In ANOVA, an interaction measures the effect of an explanatory variable (a factor) on the influence of the other explanatory variable(s) on the dependent variable. An interaction can have a non-zero value when the two explanatory variables are orthogonal, which is the situation where fraction [b] is equal to zero.

Acknowledgements

Many thanks to Pierre Legendre for critical reading and advice on this text, and help in the English translation.

Literature cited
